Protok vektorskog polja kroz hiperpovrš

$D \subseteq M^{n-1}$ - hiperpovrš

A - vekt. polje

$$I: \int_{D} A = \int_{D} \langle A, n_p \rangle \, ds = \iint_U \langle A(g(u,v)), \frac{g_u \times g_v}{\cancel{\|g_u \times g_v\|}} \rangle \cancel{\sqrt{EG-F^2}} \, du\, dv$$

ds ↳ $\sqrt{EG-F^2}$

n_p ↳ jedinični vekt. normale

U - oblast ↳ parametrizacioni skup

zbog toga se krati

$g(u,v) \to (x,y,z)$

$g_u \times g_v$

$E = \langle g_u, g_u \rangle$

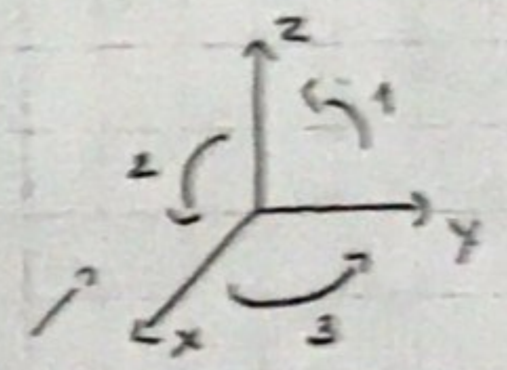
$G = \langle g_v, g_v \rangle$

$F = \langle g_u, g_v \rangle$

$\|\vec{a} \times \vec{b}\|^2 \overset{!}{=} \|a\|^2 \|b\|^2 - \langle a, b \rangle^2$

$$\Rightarrow I = \varepsilon \iint_U \langle A(g(u,v)), g_u \times g_v \rangle \, du\, dv$$

$\begin{cases} \varepsilon = 1 \\ \varepsilon = -1 \end{cases}$ u zavisnosti od toga da li je spoljašnja ili unutrašnja normala

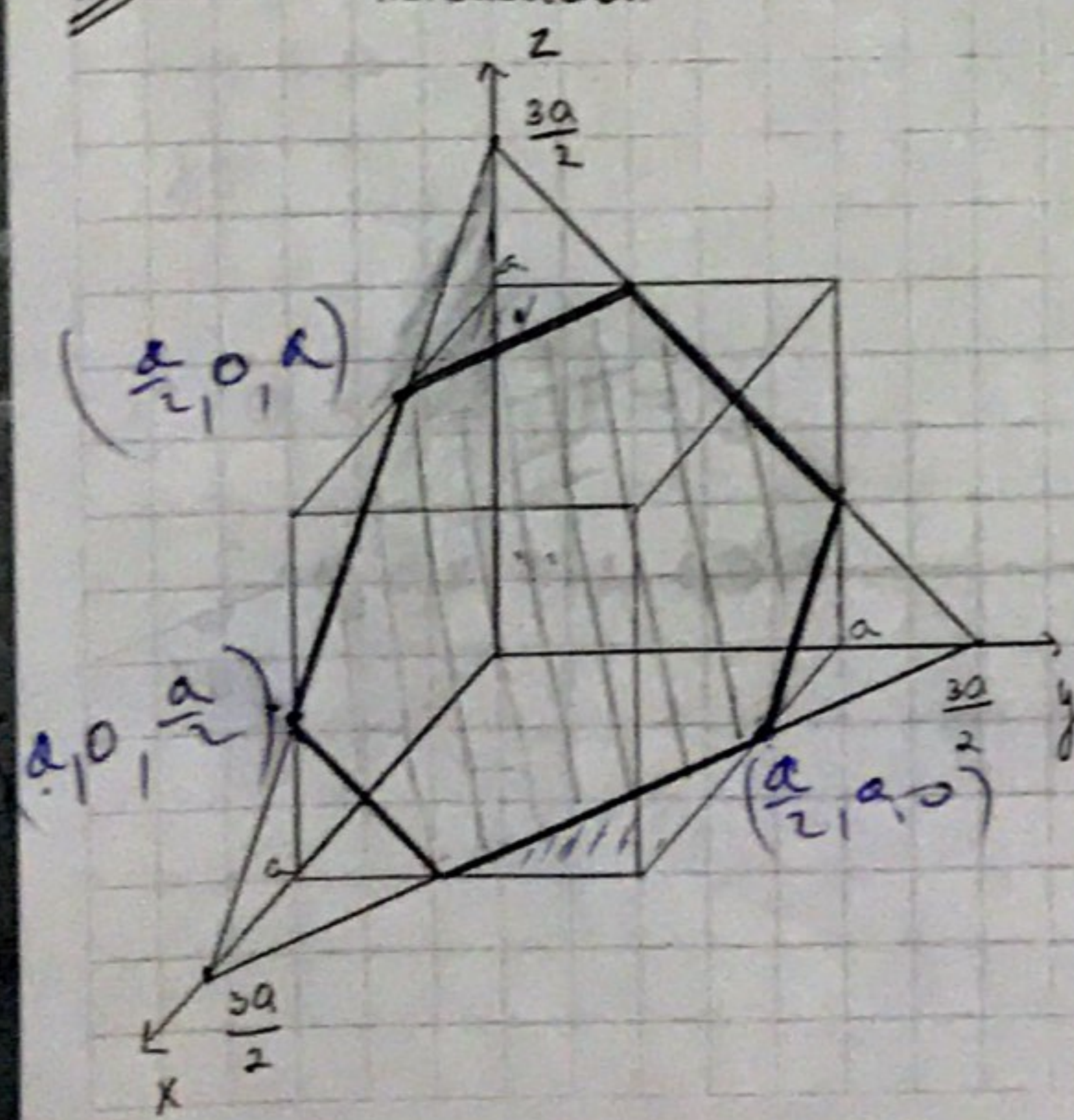


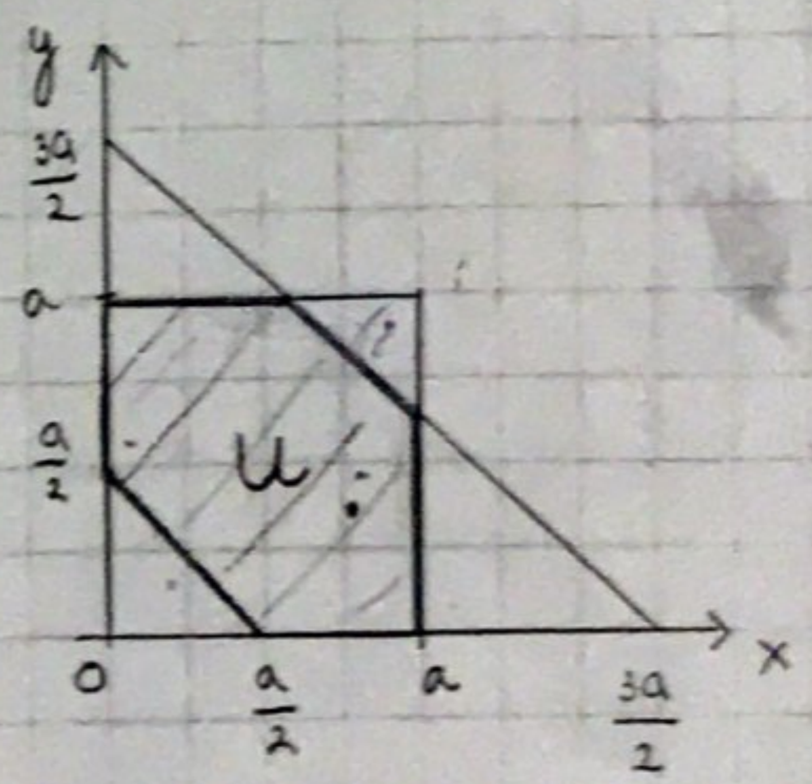
① $I = \iint_S (z+y)\, dy \wedge dz + (z+x)\, dz \wedge dx + (x+y)\, dx \wedge dy =$

S: dio ravni $x+y+z = \frac{3a}{2}$ koji se nalazi unutar $[0,a]^3$.

↳ orijentisana spoljašnjom normalom

parametrizacija: $(x,y) \overset{g}{\mapsto} (x, y, \frac{3a}{2} - x - y)$





hoćemo da predstavimo kao grafik fje pa projektujemo dole sve

$g_x = (1, 0, -1)$

$g_y = (0, 1, -1)$ $\boxed{\varepsilon = 1}$

$g_x \times g_y = (1, 1, 1)$

real. prav'la za g_u

$$\Rightarrow I = \iint_{u} \left[\left(\frac{3a}{2} - \cancel{x} - y + \cancel{y}\right)\cdot 1 + \left(\frac{3a}{2} - x - \cancel{y} + \cancel{x}\right)\cdot 1 + (\cancel{x} + \cancel{y})\cdot 1 \right] dx\,dy$$

$$= 3a \underbrace{\iint_{u} 1 \, dx\,dy}_{\mu(u)} = 3a \left(a^2 - 2\,\frac{\frac{a}{2}\cdot\frac{a}{2}}{2}\right)$$

V 14.05.2018.

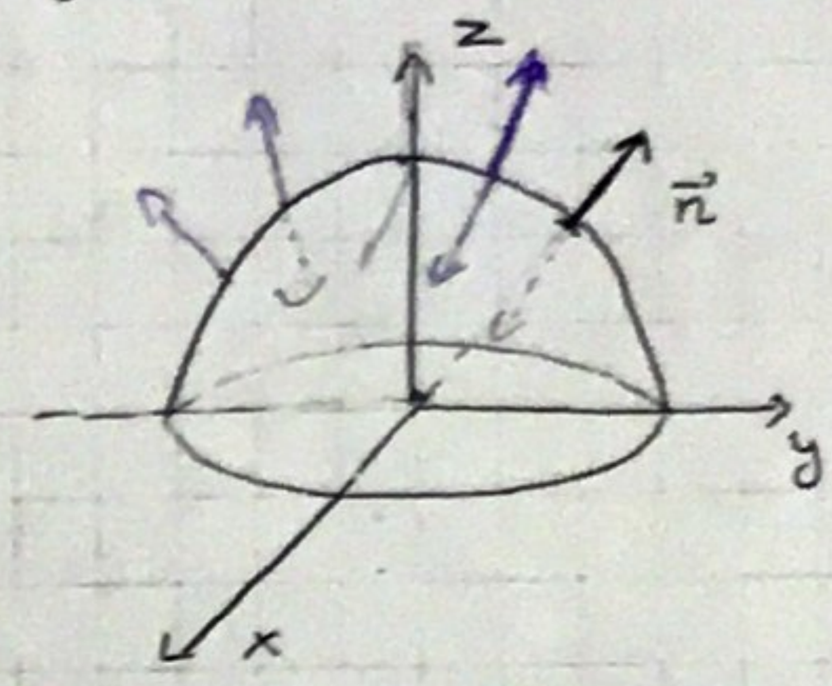
① Izračunati protok vektorskog polja $A = (x^2, y^2, z^2)$ po gornjoj jediničnoj polusferi. (ogr. po spolj. normala)

R:

parametrizujemo sferu: $(x,y) \longrightarrow (x, y, \sqrt{1-x^2-y^2})$

$$g_x = \left(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\right), \quad g_y = \left(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\right)$$

$$g_x \times g_y = \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right)$$



$$\frac{\langle \vec{e_z}, \vec{a} \rangle}{\|\vec{e_z}\| \cdot \|\vec{a}\|} = \cos \angle(\vec{e_z}, \vec{a}) \; ; \vec{e_z} = (0,0,1)$$

1° oštar $\Rightarrow \cos \angle > 0$

da bi skalarni pr. bio pozitivan

$$\int_{\vec{S}^+} A = 1 \cdot \iint_D \langle A(g(x,y)), g_x \times g_y \rangle \, dx\, dy =$$

$$= \iint_D \left(x^2 \cdot \frac{x}{\sqrt{1-x^2-y^2}} + y^2 \cdot \frac{y}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2) \cdot 1 \right) dx\, dy =$$

$$= \begin{bmatrix} x = r\cos\varphi \\ y = r\sin\varphi \end{bmatrix} = \int_0^{2\pi}\int_0^1 \Big(\underbrace{\frac{r^3\cos^3\varphi}{\sqrt{1-r^2}}}_{=0} + \underbrace{\frac{r^3\sin^3\varphi}{\sqrt{1-r^2}}}_{=0} + 1 - r^2 \Big) r\, dr\, d\varphi =$$

jer $\cos^3 x$ na $0 \rightarrow 2\pi = 0$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\, d\varphi = 2\pi \left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \frac{\pi}{2}$$

Štoksova Teorema: Neka je M^2 površ sa krajem c i pretpostavimo da su saglasno orjentisane. Neka su P, Q, R 3 realne, C^1 f-je u $\mathbb{R}^3$.

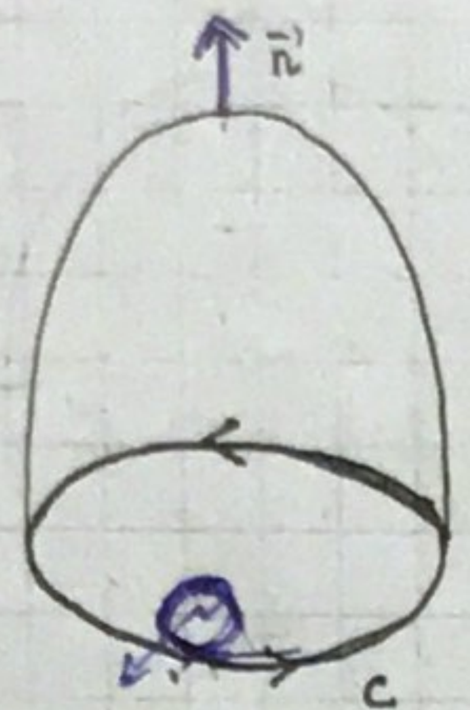
Tada važi: $\int_c P\,dx + Q\,dy + R\,dz = \iint_{M_2} \underbrace{rot(P,Q,R)}_{F} =$

Krivolinijski integral po zatvorenoj krivoj u prostoru

$$rot(P,Q,R) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \iint_{M_2} (R_y - Q_z)\,dy \wedge dz + (P_z - R_x)\,dz \wedge dx + (Q_x - P_y)\,dx \wedge dy$$

površinski integral druge vrste

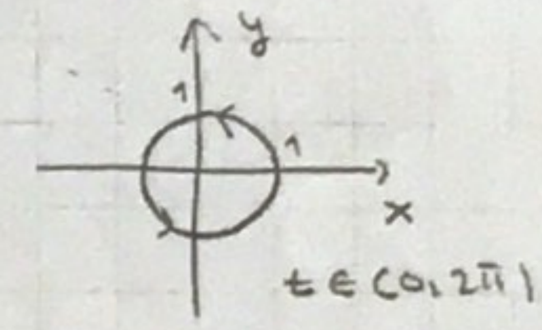
primjer:



→ kriva ove površi (saglasno orijentisana sa M^2!)

② $\int_c z\,dx + x\,dy + z\,dz$, gdje je c - jedinični krug u xOy ravni

R: $t \to (\cos t, \sin t, 0)$ parametrizacija



$$I = \int_0^{2\pi} 0 + \cos t \cos t + 0 = \int_0^{2\pi} \frac{1+\cos t}{2} = \pi$$

II način (preko štoksove teoreme):

$P = z \qquad R_y - Q_z = 0$

$Q = x \qquad P_z - R_x = 1$

$R = z \qquad Q_x - P_y = 1$

$z = 1 - x^2 - y^2 \qquad z = 0$

$$I = \iint_{S^+} 1\,dz \wedge dx + 1\,dx \wedge dy = \left(\begin{array}{l} (x,y) \to (x, y, \sqrt{1-x^2-y^2}) \\ \vec{n} = \vec{g}_x \times \vec{g}_y = \left(\frac{x}{r}, \frac{y}{r}, 1\right) \end{array} \right)$$

jer je kraj ove površi c iz zad.

$$= \iint_D \left(\frac{y}{\sqrt{1-x^2-y^2}} + 1 \right) dx\,dy = \left(\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \right) =$$

$$= \dots = \int_0^{2\pi}\int_0^1 r = \frac{1}{2} \cdot 2\pi = \pi$$

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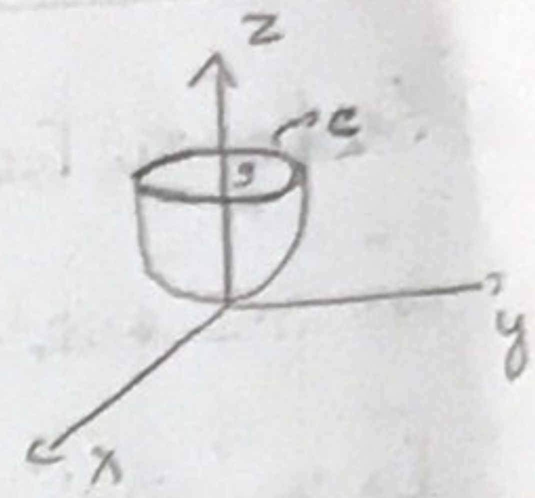
(3) $\int_C xz\,dx + x\,dy + yz\,dz$, $C: \begin{cases} x^2+y^2=z \\ z=9 \end{cases}$



(U smjeru kazaljke ako se gleda sa vrha z-ose)

I: $t \to (3\cos t, 3\sin t, 9)$

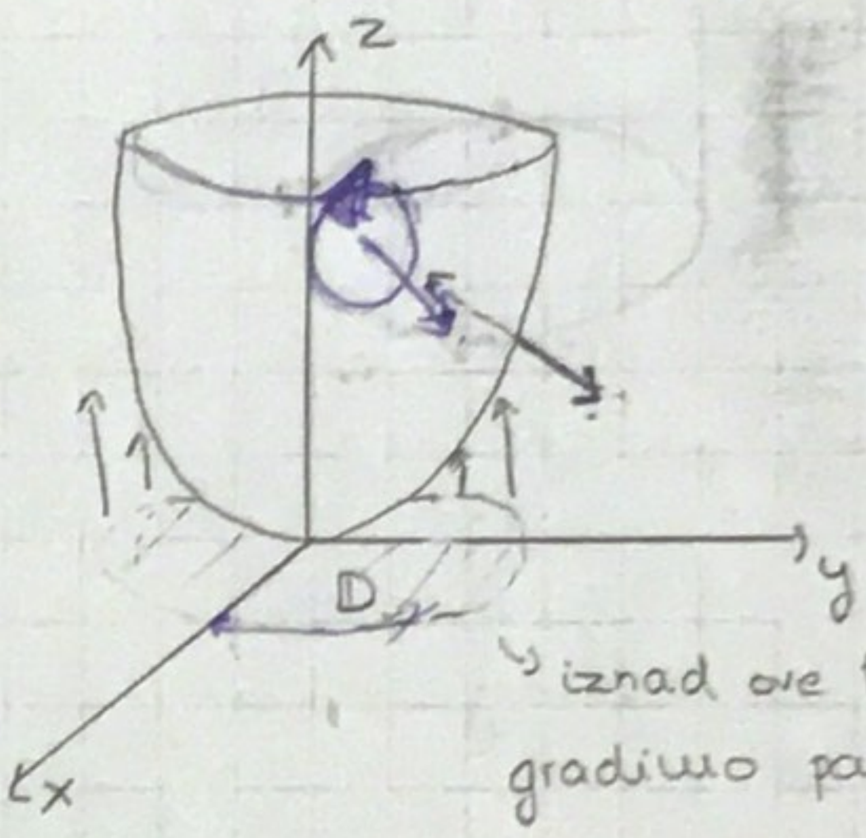
$t \in (0, 2\pi)$

$$I = \int_0^{2\pi} \left(\cancelto{0}{3\cos t \cdot 9 \cdot (-3\sin t)} + 3\cos t \cdot 3\cos t + \cancelto{0}{27\sin t \cdot (0)}\right) dt =$$

$$= \int_0^{2\pi} 9\cos^2 t \, dt = 9\pi$$

$\varepsilon = -1$ Naš rez $= -9\pi$

II način: (Štoksova T)



iznad ove kružnice gradimo paraboloid

$(x,y) \to (x, y, x^2+y^2)$

$D = B(0,3)$

↳ oblast integracije

$P = xz$ $R_y - Q_z = z = x^2+y^2$

$R = x$ $P_z - R_x = x$

$Q = yz$ $Q_x - P_y = 1$

$g_x = (1, 0, 2x)$

$g_y = (0, 1, 2y)$

$g_x \times g_y = (-2x, -2y, 1)$

$\varepsilon = -1$ (ugao koji $\vec{n}$ koji nama treba gradi tupi ugao sa z-osom $\Rightarrow$ treća koo. treba da bude negativna)

$$I = (-1)\iint_D \left((-2x)(x^2+y^2) + x(-2y) + 1\right) dxdy = \ldots \text{polarne koo} \ldots = -9\pi$$

Razlika je što nismo uzeli dobru orijentaciju u I dijelu.

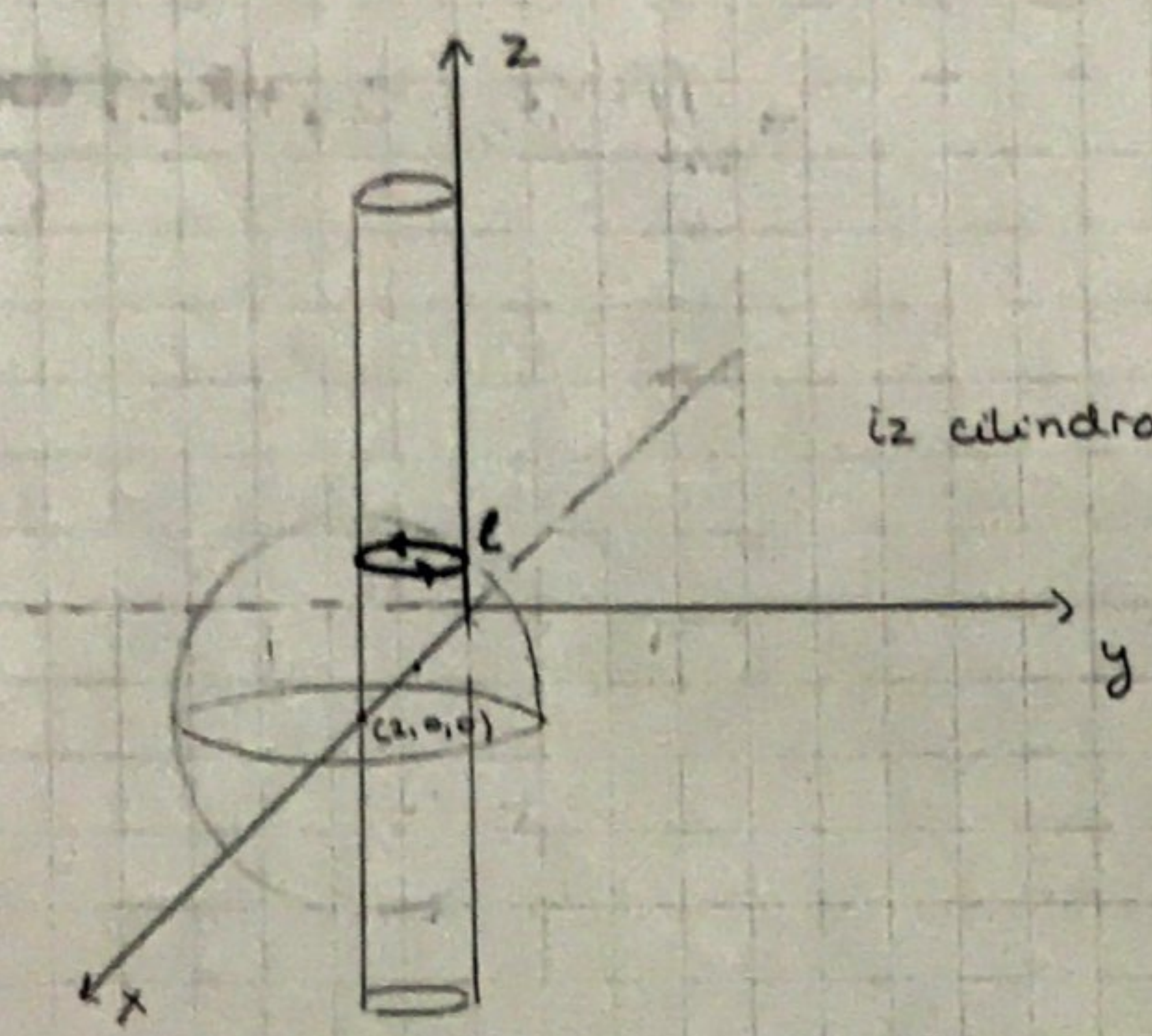
4. $\int\limits_{\ell} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2+y^2)dz$,

gdje je $\ell: \begin{cases} x^2+y^2+z^2 = 4x \\ x^2+y^2 = 2x \\ z>0 \end{cases}$, gledamo sa vrha z-ose (pozitivno orijentisana)

R:

$x^2+y^2+z^2 = 4x \iff (x-2)^2+y^2+z^2 = 4$ sfera

$x^2+y^2 = 2x \iff (x-1)^2+y^2 = 1$ cilindar



iz cilindra $\begin{cases} x-1 = \cos t \Rightarrow x = 1+\cos t \\ y = \sin t \end{cases}$

$z = \sqrt{4-(x-2)^2-y^2}$

$z = \sqrt{4-(\cos t-1)^2-\sin^2 t}$

$t \longrightarrow (\cos t + 1, \sin t, \sqrt{4-(\cos t-1)^2-\sin^2 t})$

$t \in (0, 2\pi)$ $\quad \varepsilon = 1$

II način (Štoksova):

$(x,y) \longrightarrow (x, y, \sqrt{4-(x-2)^2-y^2})$

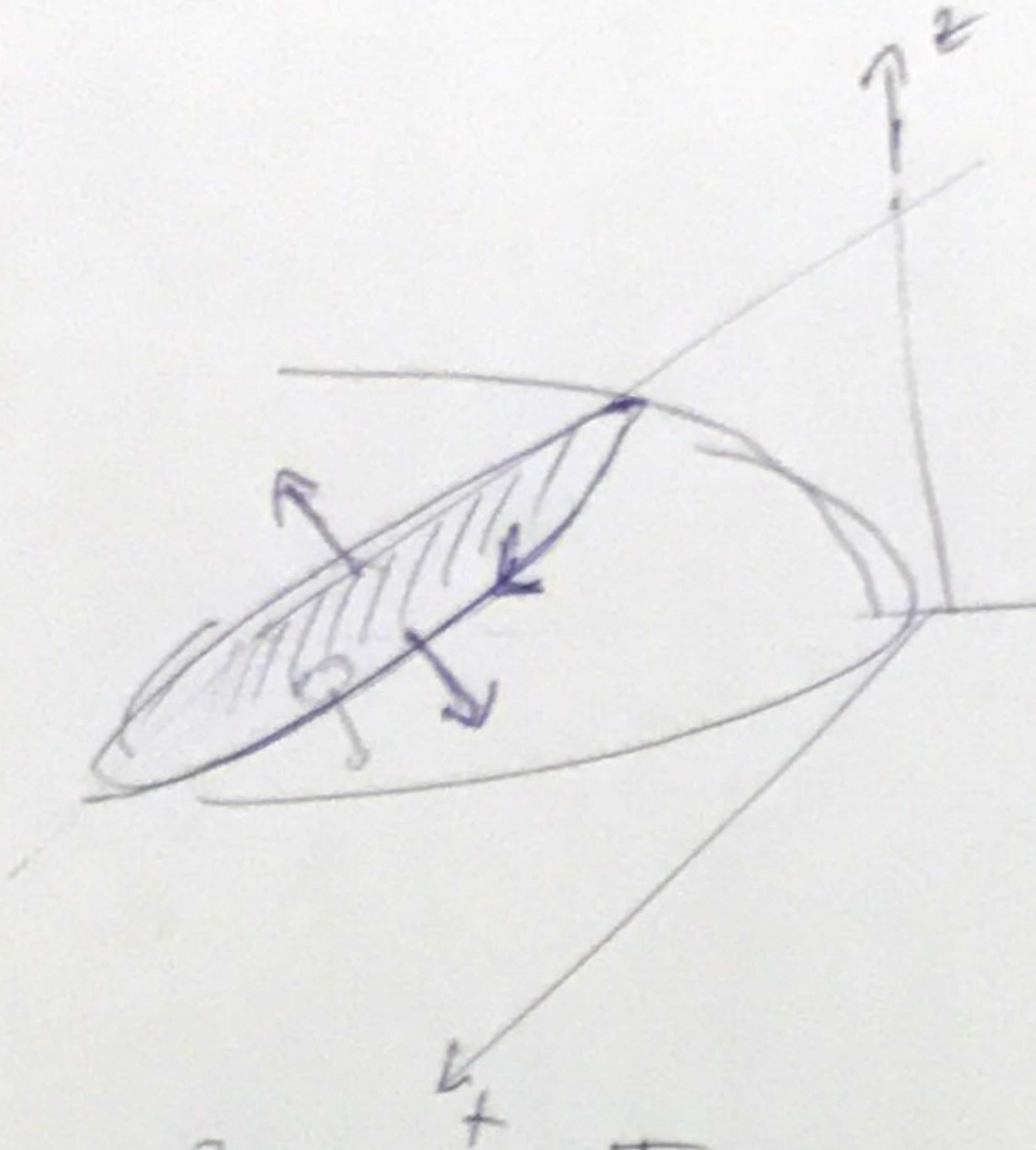
$\vec{n} = \vec{g}_x \times \vec{g}_y$

(Zad)

$A = (y^2, z^2, x^2)$

$C: \begin{cases} y + x^2 + z^2 = 0 \\ y = 2z - 8 \end{cases}$

poz orijentaciju gledamo sa poz dijela y-ose

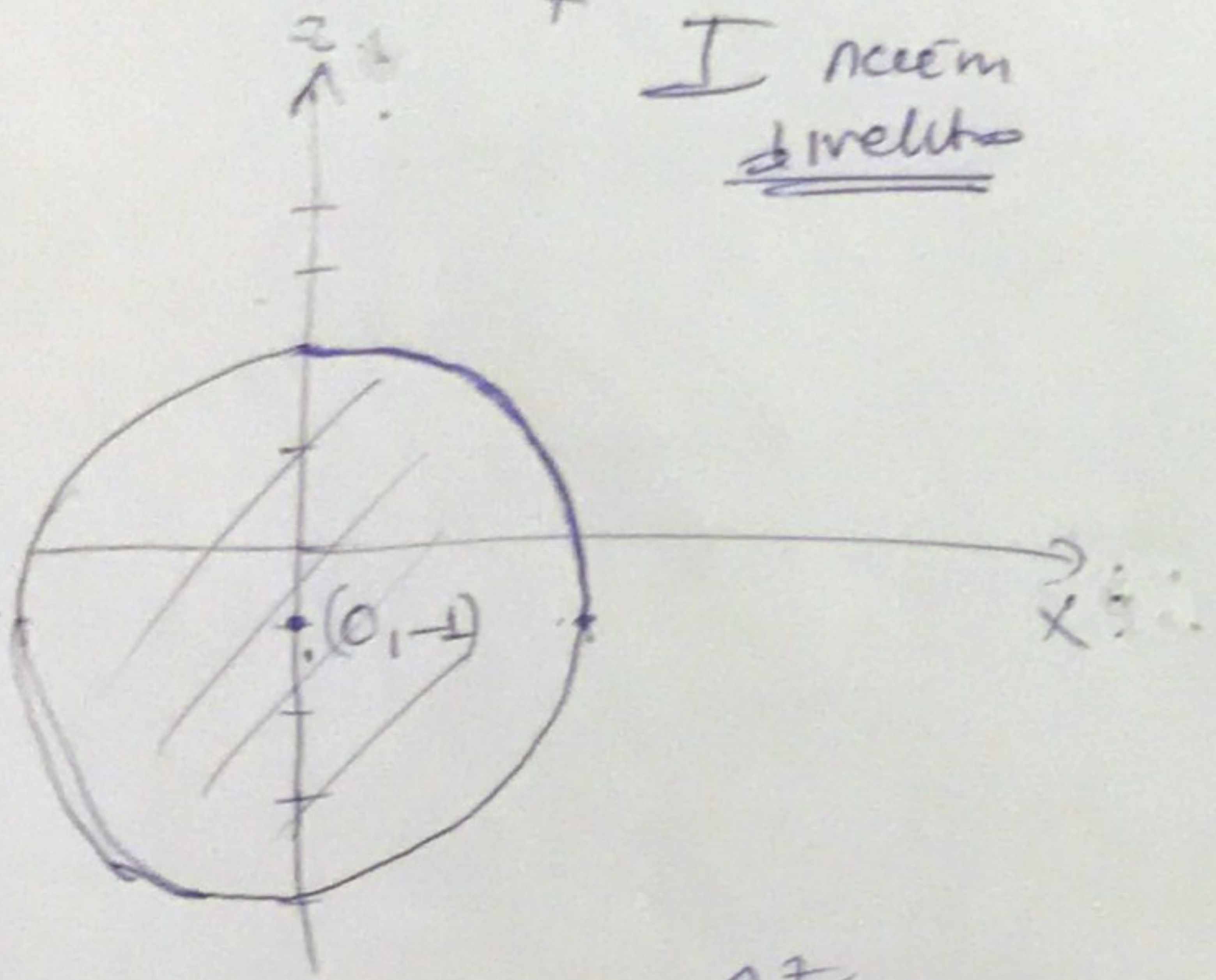


$2z - 8 = -x^2 - z^2$

$x^2 + z^2 + 2z = 8$

$x^2 + (z+1)^2 = 3^2.$

I način direktno

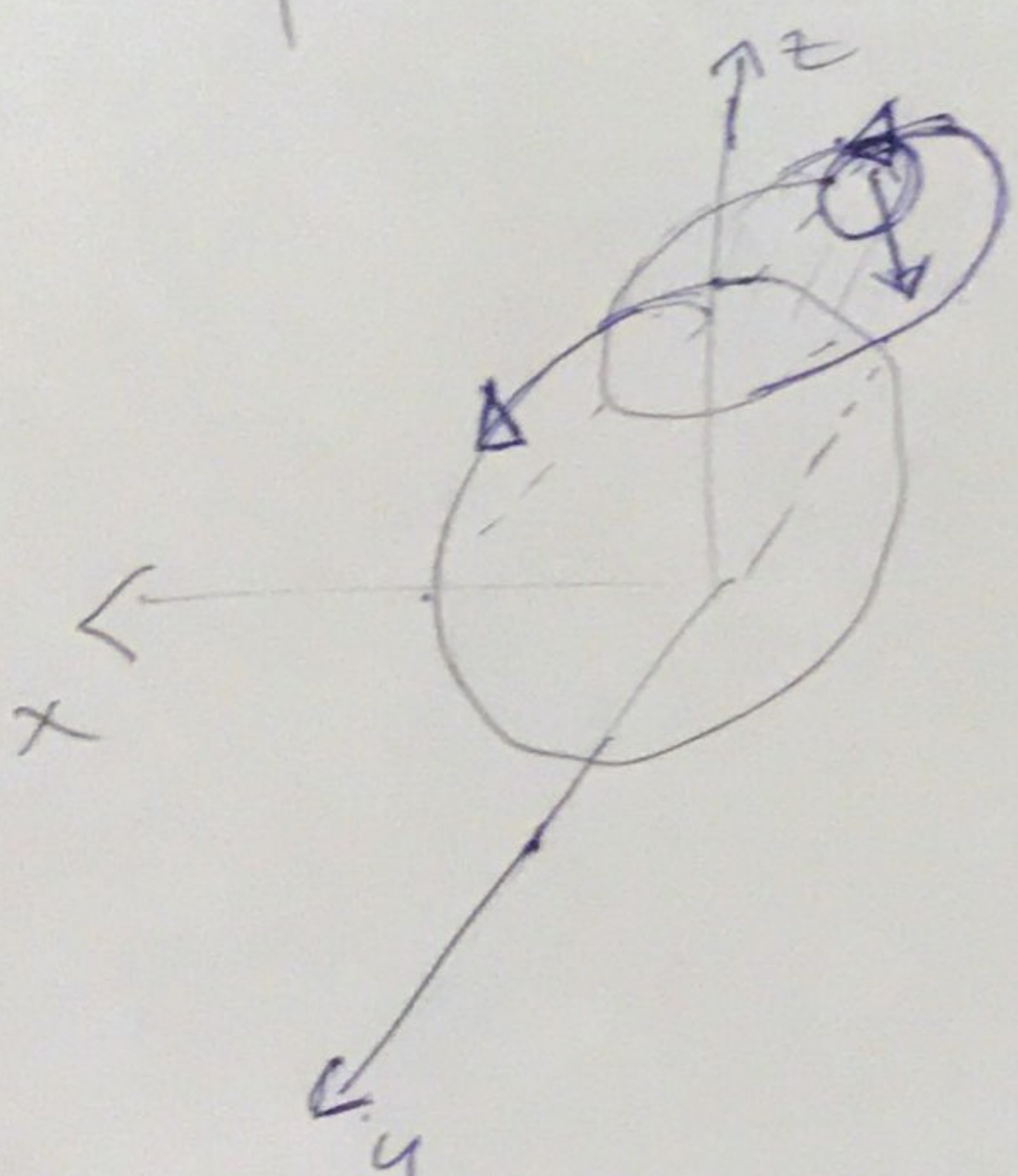


$t \to (3\cos t, 6\sin t - 10, -1 + 3\sin t)$

$\left.\begin{array}{l} x = 3\cos t \\ z = -1 + 3\sin t \end{array}\right\} t \in (0, 2\pi)$

$y = -2 + 6\sin t - 8$
$= 6\sin t - 10$

$\varepsilon = -1$



$$I = -\int_0^{2\pi} \Big[(6\sin t - 10)^2 (-3\sin t) + (-1 + 3\sin t)^2 (6\cos t) + (3\cos t)^2 \cdot (3\cos t) \Big] dt$$

$= -360\pi$

II пачин

$(x,z) \xrightarrow{g} (x, 2z-8, z)$ $\quad U: x^2+(z+1)^2 \leq 3^2$

$g_x = (1,0,-)$ $\quad g_x \times g_z = (0,-1,2)$

$g_z = (0,2,1)$ $\quad \varepsilon = -1$

$P = y^2$

$Q = z^2$

$R = x^2$

$R_y - Q_z = -2z$

$P_z - R_x = -2x$

$Q_x - P_y = -2y$

$$\iint_M (-2z)\,dy\wedge dz + (-2x)\,dz\wedge dx + (-2y)\,dx\wedge dy$$

$$\overset{\varepsilon=-1}{=} +2\iint_U \cancel{-2z\cdot 0}\left(z\cdot 0 + x\cdot(-1) + \overset{2z-8}{\overset{\parallel}{y}}\cdot(+2)\right) dx\,dz$$

$$= 2\iint_U (-x+4z-16)\,dx\,dz = 2\iint_U -16-x+4z\;dx\,dz =$$

$$= \cancel{2\cdot 16}\cdot \left[\begin{matrix} x = r\cos\varphi \\ z = -1+r\sin\varphi \end{matrix}\right] = -360\pi$$

$r \in (0,3)$

$\varphi \in (0,2\pi)$

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$\|a\times b\|^2 = \|a\|^2\|b\|^2 \cdot \sin^2\angle(a,b) =$

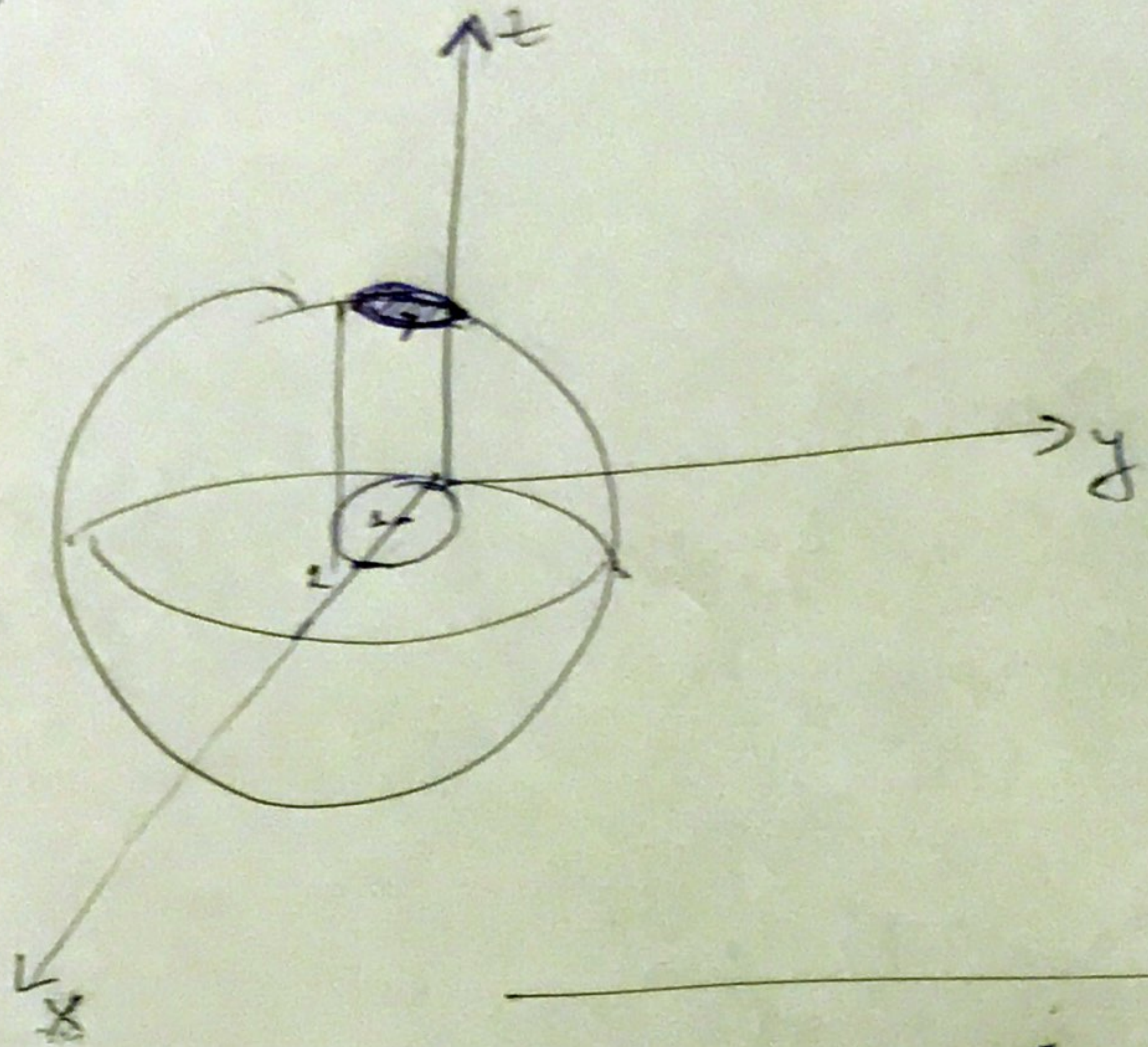
$= \|a\|^2\|b\|^2 - \|a\|^2\|b\|^2\cos^2\angle(a,b)$

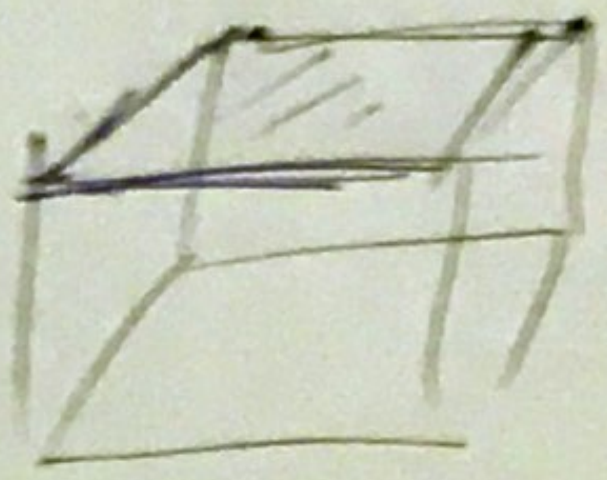
$= \|a\|^2\|b\|^2 - \langle a,b\rangle^2 \quad /\sqrt{\ }$

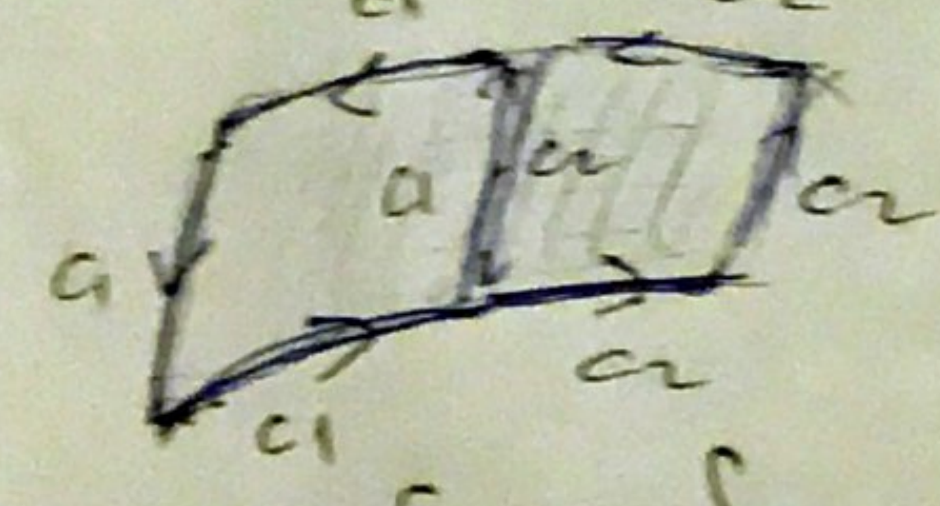
$a = g_u, \quad b = g_v$

$\|\vec{g_u}\times\vec{g_v}\| = \sqrt{\langle g_u,g_u\rangle\langle g_v,g_v\rangle - \langle g_u,g_v\rangle^2} = \sqrt{EG-F^2} = G(g_u,g_v)$

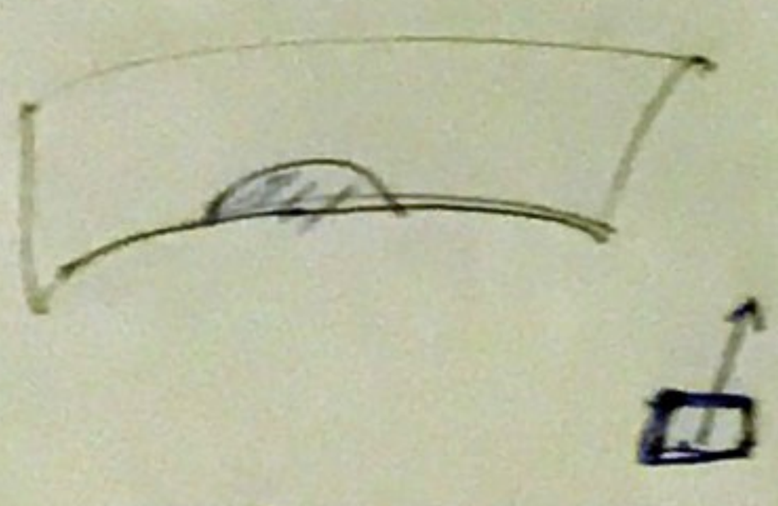
6

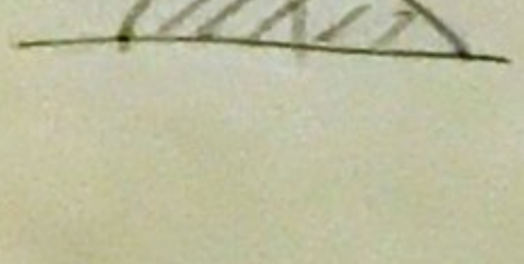


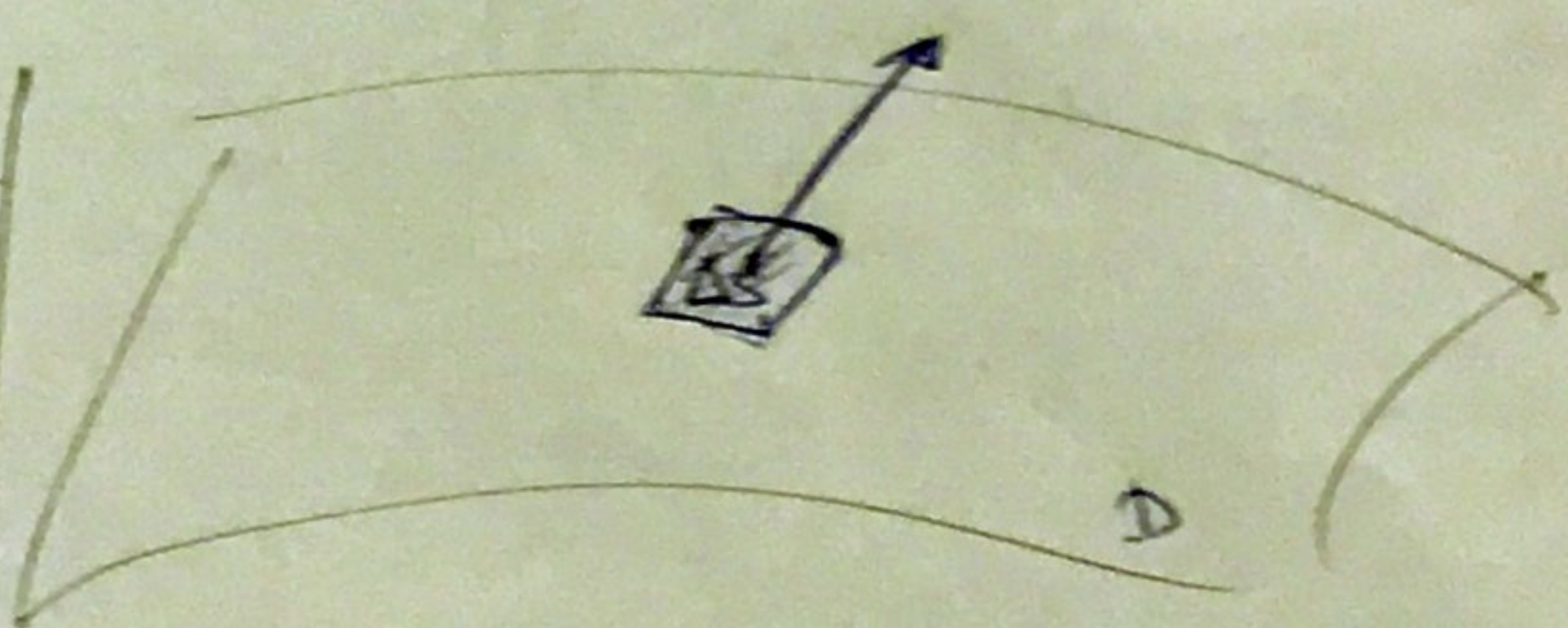


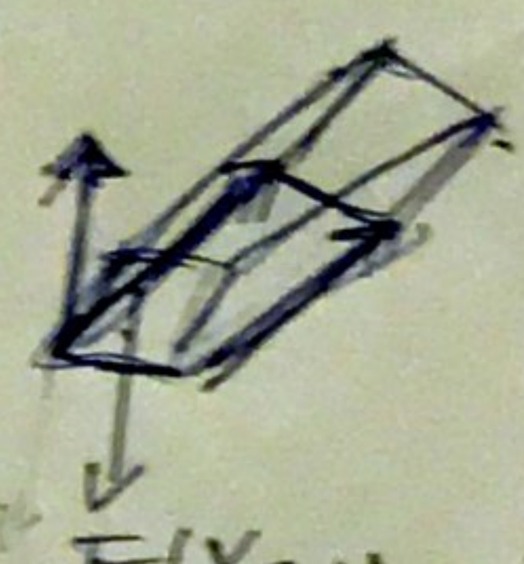


$\int_{c_1\cup c_2} = \int_{c_1} + \int_{c_2}$









$F(x_0,y_0,z_0)$

$F \; \langle F,n\rangle\cdot\Delta S$

$F(x_0,y_0,z_0)$